③ $\exists$ p-adic analogue of Conj, Th, Rem ①, ②

Furusho-Y

Conj (isobar conj)

$$\sum_{n\geq 0} Z_n = \bigoplus_{n\geq 0} Z_n$$

$\mathbb{Q}$-span in $\mathbb{R}$

Rem If isobar conj. holds, for $\forall x$: MZV of wt >0

$\forall N$, $\underbrace{x^N, x^{N-1}, \ldots, x, 1}_{\text{no } \mathbb{Q}\text{-linear rel'n}}$

$\Rightarrow x$: transcendental!

(In particular, $\zeta(3), \zeta(5), \zeta(7), \ldots$ are transcendental!)

relations today

- double shuffle rel'n
$\Uparrow$
- associator rel'ns (cycle rel'ns)
$\Uparrow$
- K-theoretic rel'n

... lin. rel'n among

Rem ① $\exists$ "twisted version" (multiple

the K-theoretic

holds,

ZV of wt > 0

..., x, 1

Q-linear rel'n

transcendental!

In particular,
$\zeta(3), \zeta(5), \zeta(7), \ldots$
are transcendental!

relations   today

- double shuffle rel'ns ← explicit
$\Uparrow$
- associator rel'ns
  (cycle rel'ns)
$\Uparrow$
- K-theoretic rel'n ← strongest (not explicit)

(↔ KRV)

DMR ← Alekseev-Torossian

$\bigcup$ ← Furusho

GRT

$\bigcup$ ← Brown

$Gal(MT(\mathbb{Z}), \omega)$

§1. double shuffle rel'n

§1.1 multiple polylog & integral expression

naughtily $\frac{1}{1-T} = 1 + T + T^2 + T^3 + \cdots$

$\int_0^T \frac{(\cdots)}{T} dT$

$1 + T + \frac{T^2}{2} + \frac{T^3}{3} + \cdots \xmapsto{T=1} \zeta(1) \quad (\infty)$

$\int_0^T \frac{(\cdots)}{T} dT$

$1 + T + \frac{T^2}{2^2} + \frac{T^3}{3^2} + \cdots \xmapsto{T=1} \zeta(2)$

$\int_0^T \frac{(\cdots)}{T} dT$

$1 + T + \frac{T^2}{2^3} + \frac{T^3}{3^3} + \cdots \xmapsto{T=1} \zeta(3)$

$\int_0^T \frac{(\cdots)}{1-T} dT$

$\sum_{0<n<m} \frac{T^m}{n^3 m} \xmapsto{T=1} \zeta(3,1) \quad (\infty)$

$\int_0^T \frac{(\cdots)}{T} dT$

$\sum_{0<n<m} \frac{T^m}{n^3 m^2} \xmapsto{T=1} \zeta(3,2)$

multiple polylog

$Li_{k_1, \ldots, k_d}$

Lem $\frac{d Li_{k_1, \ldots}}{dz}$

$(Li_1(z) = \sum \frac{z^n}{n} = -\log \ldots$

$\zeta(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$

Rem If isobar conj. holds,

for $\forall x$: MZV of wt >0

$\forall N$, $\underbrace{x^N, x^{N-1}, \ldots, x, 1}_{\text{no } \mathbb{Q}\text{-linear rel'n}}$

$\Rightarrow x$: transcendental !

relations

- double shuffle
- associator
- (cycle …
- K-theoretic

$\overset{T=1}{\longmapsto} \zeta(3)$

$(, \infty)$

$\longmapsto \zeta(3,1)$

$\overset{T=1}{\longmapsto} \zeta(3,2)$

<u>multiple polylog</u> ($d=1$: polylog)

$$Li_{k_1,\dots,k_d}(z) = \sum_{n_1<\dots<n_d} \frac{z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}} \quad (|z|<1)$$

<u>Lem</u>

$$\frac{d\,Li_{k_1,\dots,k_d}(z)}{dz} = \begin{cases} \frac{1}{z} Li_{k_1,\dots,k_d-1}(z) & \text{if } k_d>1 \\ \frac{1}{1-z} Li_{k_1,\dots,k_{d-1}} & \text{if } k_d=1,\ d>1 \\ \frac{1}{1-z} & \text{if } k_d=1,\ d=1 \end{cases}$$

$\left(Li_1(z) = \sum \frac{z^n}{n} = -\log(1-z) \right)$

(an. cont. & $z=1$ $\longmapsto \zeta(k_1,\dots,k_d)$)

$$\begin{array}{ccccc} 1 \to U & \to & \pi_1(\mathbb{P}^1\setminus\{0,1,\infty\}) & \to & \pi_1(\mathbb{P}^1\setminus\{0,\infty\}) \to 1 \\ & & & & \| \\ \downarrow & & \downarrow & & \mathbb{Q}(1) \\ & & & & \| \\ 1 \to U^{ab} & \to & Pol(1) & \to & \mathbb{Q}(1) \to 1 \\ \| & & & & \\ Log(1) & & \text{polylog ext'n.} & & \end{array}$$

<u>relations</u> <u>today</u>

- double shuffle rel'ns ← explicit
- ⇑
- associator rel'ns ←
- ⇑ (cycle rel'ns)

(⟷ <u>KRV</u>)

Alekseev–Torossian

<u>DMR</u>

$\Downarrow$ ← Furusho

<u>GRT</u>

$\Downarrow$ ← Brown

$\Downarrow$

$\subset$ D(MT(Q), ω)

① $(LHS) = \sum_{n_1<\cdots<n_d} \frac{z^{n_d-1}}{n_1^{k_1}\cdots n_d^{k_d-1}}$

$$= \begin{cases} \frac{1}{z} Li_{k_1,\ldots,k_d-1}(z) & k_d>1 \\ \sum_{n_1<\cdots<n_{d-1}} \frac{1}{n_1^{k_1}\cdots n_{d-1}^{k_{d-1}}} \underbrace{\sum_{n_d=n_{d-1}+1}^{\infty} z^{n_d-1}}_{\frac{z^{n_{d-1}}}{1-z}} & \text{if } k_d=1 \end{cases}$$

//

Cor ∃ an. cont. (as multi-valued fct)

$$\zeta(k_1,\ldots,k_d) = \underbrace{\int_0^1 \frac{dT}{T}\int_0^T \frac{dT}{T}\cdots\int_0^T \frac{dT}{T}}_{k_d-1}\int_0^T \frac{dT}{1-T}\underbrace{\int_0^T \frac{dT}{T}\cdots\int_0^T \frac{dT}{T}}_{k_{d-1}-1}$$

$$\cdots\cdots \underbrace{\int_0^T \frac{dT}{T}\cdots\int_0^T \frac{dT}{T}}_{k_1-1}\int_0^T \frac{dT}{1-T}$$

symbol $a := \int_0^T \frac{(\cdots)}{T} dT$

$b := \int_0^T \frac{(\cdots)}{1-T} dT$

$\zeta(k_1,\ldots,k_d) = a^{k_d-1} b$

$+ \frac{T^2}{2^3} + \frac{T^3}{3^3} + \cdots +$

$\int_0^T \frac{(\cdots)}{1-T} dT$ (∞)

$\sum_{n<m} \frac{T^m}{n^3 m} \xrightarrow{T=1} \zeta(3,1)$

multiple polylog

$Li_{k_1,\ldots,k_d}(z) := \sum_{n_1<\cdots}$

Lem $\frac{d\,Li_{k_1,\ldots,k_d}(z)}{dz} = \begin{cases} \frac{1}{z}\cdots \\ \frac{1}{1-\cdots} \end{cases}$

Cor ∃ an. cont. (as multi-valued f(x))

$$\zeta(k_1,\dots,k_d) = \underbrace{\int_0^1\frac{dT}{T}\int_0^T\frac{dT}{T}\cdots\int_0^T\frac{dT}{T}}_{k_d-1}\int_0^T\frac{dT}{1-T}\underbrace{\int_0^T\frac{dT}{T}\cdots\int_0^T\frac{dT}{T}}_{k_{d-1}-1}$$

$$\cdots\cdots\underbrace{\int_0^T\frac{dT}{T}\cdots\int_0^T\frac{dT}{T}}_{k_1-1}\int_0^T\frac{dT}{1-T}$$

$\int_0^T \frac{(\dots)}{T}dT$ increments the last index

$\int_0^T \frac{(\dots)}{1-T}dT$ increments the depth w/ last index = 1

$$\dots 1 \quad \frac{T^2}{2^3}+\frac{T^3}{3^3}+\cdots+$$

$$\int_0^T \frac{(\dots)}{1-T}dT \qquad (\infty)$$

$$\frac{T^m}{\dots} \qquad \xrightarrow{T=1} \zeta(3,1)$$

multiple polylog (d=1 polylog)

$$Li_{k_1,\dots,k_d}(z) = \sum_{n_1<\cdots<n_d}\frac{z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}} \qquad (|z|<1)$$

Lem $\frac{d\,Li_{k_1,\dots,k_d}(z)}{dz} = \begin{cases}\frac{1}{z}Li_{k_1,\dots,k_d-1}(z) & \text{if } k_d>1\\ \dots\end{cases}$ $\left(\begin{array}{l}\text{an. cont. \& } z=1\\ \mapsto \zeta(k_1,\dots,k_d)\end{array}\right)$

symbol $\quad a := \int_0^T \frac{(\cdot)}{T}\, dT$

$\qquad\quad b := \int_0^T \frac{(\cdot)}{1-T}\, dT$

$\zeta(k_1,\dots,k_d) = a^{k_d-1} b \cdots a^{k_1-1} b \Big|_{T=1}$

the # of b's = depth

the total # of a, b's = weight

§1.2 duality

e.g.

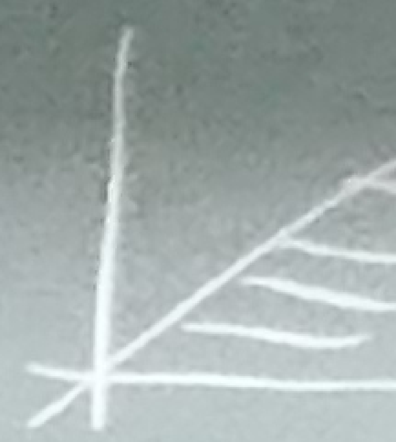
$$\zeta(3) = \int_0^1 \frac{dT}{T} \int_0^T \frac{dT}{T} \int_0^T \frac{dT}{1-T}$$

$$= \int_1^0 \frac{-dT}{1-T} \int_1^{1-T} \frac{-dT}{1-T} \int_1^{1-T} \frac{-dT}{T}$$

change $T \mapsto 1-T$

$$= \int_0^1 \frac{dT}{1-T} \int_{1-T}^1 \frac{dT}{1-T} \int_{1-T}^1 \frac{dT}{T}$$

change the order



$$= \int_0^1 \frac{dT}{T} \int_0^{T} \cdots$$

$$= \zeta(1,2)$$

e.g. $\zeta($

1,2 duality

$$\zeta(3) = \int_0^1 \frac{dT}{T}\int_0^T \frac{dT}{T}\int_0^T \frac{dT}{1-T}$$

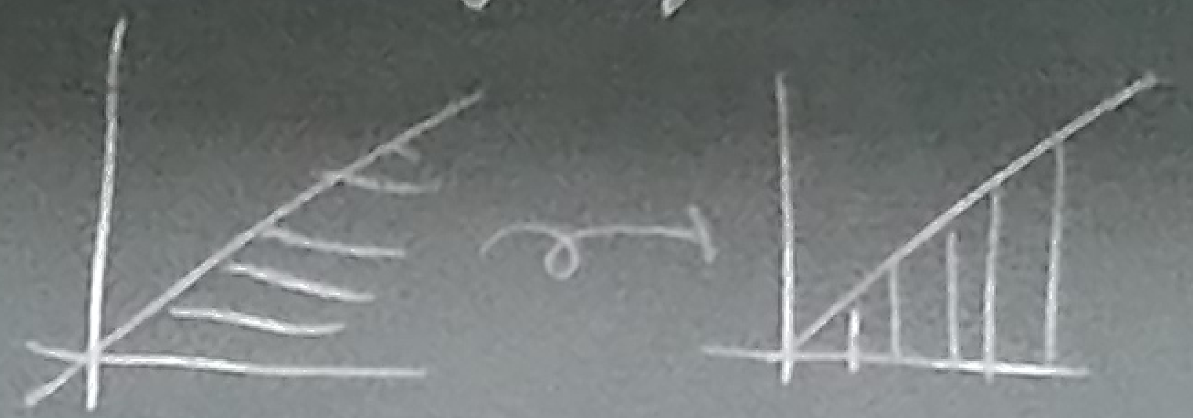
$$= \int_1^0 \frac{-dT}{1-T}\int_1^{1-T} \frac{-dT}{1-T}\int_1^{1-T} \frac{-dT}{T}$$

...ge

$T \longmapsto 1-T$

$$= \int_0^1 \frac{dT}{1-T}\int_{1-T}^1 \frac{dT}{1-T}\int_{1-T}^1 \frac{dT}{T}$$

change the order of integration



$$= \int_0^1 \frac{dT}{T}\int_0^T \frac{dT}{1-T}\int_0^T \frac{dT}{1-T}$$

$$= \zeta(1,2)$$

$\exists!$ $(\ )^* : \mathrm{Word}(a,b) \longrightarrow \mathrm{Word}(a,b)$

anti-isom

$\left((ww')^* = (w')^* w^* \right)$

s.t. $a^* = b$

$b^* = a$

duality $\forall w \in a\,\mathrm{Word}(a,b)\,b$

$$w\big|_{T=1} = (w)^*\big|_{T=1}$$

(symmetry of $\mathbb{P}^1 \setminus \{0,1,\infty\}$

$T \leftrightarrow 1-T$)

$\int_0^T \frac{(\ldots)}{T} dT$ increments the last index

$\int_0^T \frac{(\ldots)}{1-T} dT$ increments the depth w/ last index $=1$

e.g. $\zeta(4) = a^3 b|_{T=1} = ab^3|_{T=1}$

$= \zeta(1,1,2)$

§1.3 double shuffle rel'n

e.g.

$$\zeta(2)\zeta(2) = \sum_{n>0} \frac{1}{n^2} \sum_{m>0} \frac{1}{m^2}$$

$$= \sum_{n<m} + \sum_{m<n} + \sum_{m=n>0} = 2\zeta(2,2) \ldots$$

series shuffle

(harmonic shuffle)

$$\int_{0<x_1<x_2<1} \frac{dx_1}{x_1} \frac{dx_2}{1-x_2} \cdot \int_{0<y_1<y_2<1} \frac{dy_1}{y_1} \frac{dy_2}{1-y_2}$$

$(Z_n Z_m \ldots$

we define inductively

$a \cdot b$, $w, w' \in$

change ...

e.g.

$$\zeta(3) = \int_0^1 \frac{dT}{T} \int_0 \frac{\;}{T} \int_0 \frac{\;}{1-T}$$

$$= \int^0 \frac{-dT}{\;} \int^{1-T} \frac{-dT}{\;} \int^{1-T} \ldots$$

e.g.

$$\zeta(2)\zeta(2) = \sum_{n>0}\frac{1}{n^2}\sum_{m>0}\frac{1}{m^2} \qquad \left(\mathbb{Z}_n\,\mathbb{Z}_m \subset \mathbb{Z}_{n+m} \right)$$

$$= \sum_{n<m} + \sum_{m<n} + \sum_{m=n>0} = 2\zeta(2,2)+\zeta(4)$$

series shuffle (harmonic shuffle)

$$\|$$

$$\int_{0<x_1<x_2<1}\frac{dx_1}{x_1}\frac{dx_2}{1-x_2}\cdot\int_{0<y_1<y_2<1}\frac{dy_1}{y_1}\frac{dy_2}{1-y_2}$$

$$\frac{dx_1}{x_1}\frac{dx_2}{1-x_2}\frac{dy_1}{y_1}\frac{dy_2}{1-y_2}$$

$$ab \sqcup\!\sqcup ab = a(b \sqcup\!\sqcup ab) + a(ab \sqcup\!\sqcup b) = 2a(ab \sqcup\!\sqcup b) = 2abab + 4aabb$$

integral shuffle

$$= \int_{0<x_1<x_2<y_1<y_2<1} + \int_{0<x_1<y_1<x_2<y_2<1} + \int_{0<y_1<x_1<x_2<y_2<1} + \int_{0<x_1<y_1<y_2<x_2<1} + \int_{0<y_1<x_1<y_2<x_2<1} + \int_{0<y_1<y_2<x_1<x_2<1}$$

$$= 2\zeta(2,2)+4\zeta(1,3)$$ prism decomp. of the integration domain

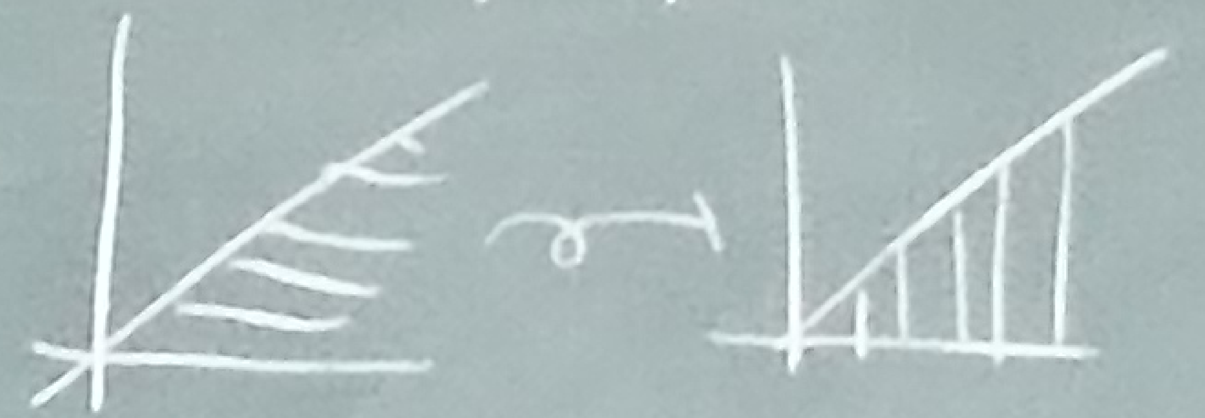
$$\Rightarrow \zeta(4) = 4\zeta(1,3)$$

e.g.

$$\zeta(3) = \int_0^1\frac{dT}{T}\int_0^{T}\frac{dT}{T}\int_0^{T}\frac{dT}{1-T}$$

$$= \int_1^0\frac{-dT}{1-T}\int_1^{1-T}\frac{-dT}{1-T}\int_1^{1-T}\frac{-dT}{T}$$

change the order of integration



$$= \int_0^1\frac{dT}{T}\int_0^T\frac{dT}{T}\int_0^T\frac{dT}{1-T}$$

$\exists!\ (\ )^* : \mathrm{Word}(a,b) \to \mathrm{Word}(a,b)$

anti-isom.

$\left((ww')^* = (w')^* w^* \right)$

s.t. $a^* = b$